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On the S-matrix of type-0 string theory

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Abstract: The recent discovery of non-perturbatively stable two-dimensional string backgrounds and their dual matix models allows the study of complete scattering matrices in string theory. In this note we adapt work of Moore, Plesser, and Ramgoolam on the bosonic string to compute the exact S-matrices of 0A and 0B string theory in two dimensions. Unitarity of the 0B theory requires the inclusion of massless soliton sectors carrying RR scalar charge as asymptotic states. We propose a regularization of IR divergences and find transition probabilities that distinguish the otherwise energetically degenerate soliton sectors. Unstable D-branes can decay into distinct soliton sectors.

Keywords: String Duality, Matrix Models.

Contents

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1. Introduction

Matrix model duals of string theory in low dimensions were extensively studied a decade ago (see [1, 2, 3] and references therein). According to the $c = 1$ version of this duality, strings in two dimensions are equivalent to free fermions in an inverted harmonic oscillator potential. In particular, the linear dilaton 'vacuum' of bosonic string theory with asymptotic string coupling g corresponds, in the dual picture, to a Fermi sea which is filled on one side of the potential up to a distance $\mu \sim (g\sqrt{\alpha'})^{-1}$ from the top (see figure 1).

The most important lesson learned from the study of these matrix models has to do with the strength of non-perturbative effects in string theory. The matrix model has obvious non-perturbative effects of order $e^{-\mathcal{O}(\sqrt{\alpha'}\mu)} \sim e^{-\mathcal{O}(1/g)}$ due to the tunneling of a single fermion through the potential barrier to the other side. Effects of this order, as

Figure 1: Matrix model description of the linear dilaton 'vacuum' of two-dimensional bosonic string theory (left), and the stable vacuum of the two-dimensional 0B string theory (right).

opposed to the smaller $e^{-\mathcal{O}(1/g^2)}$ effects familiar from gauge theories, were found to be a general feature of string theory even in higher dimensions [4], appearing naturally when worldsheets with boundaries are included [5].

Although these developments helped lay the groundwork for the second superstring revolution, the discovery of D-branes, and ultimately the gauge theory/gravity dualities exemplified by the AdS/CFT correspondence, the matrix model itself went into hibernation for a good reason: the bosonic model is obviously unstable. Although there is no tachyon in its perturbative spectrum, the theory of the half-filled Fermi sea is non-perturbatively unstable against tunneling through the barrier to fill up the other half of the sea.

Now ten years later, our growing comfort with tachyon condensation in string theory has led, following [6], to a $c = 1$ matrix model renaissance (some related work includes [7]-[19]). In particular, the endpoint of the aforementioned instability, in which both sides of the Fermi sea are filled up to the same level, has been identified[1] as the unique stable vacuum of two-dimensional type-0B string theory [12, 13], with the matrix model reinterpreted as the dual open string dynamics of a collection of unstable D-particles [6]. This remarkable proposal gives us a simple and useful calculational framework for studying a complete, non-perturbatively well-defined string theory.

The exact spacetime solution corresponding to the double-filled Fermi sea is not known; indeed, even the action for which it is a solution is unknown.[2] Qualitatively it resembles the linear dilaton solution of the bosonic theory in that there is an asymptotically flat region in one direction where the coupling vanishes, while the strong-coupling region in the other direction is cut off by a "Liouville wall," a condensation of the massless "tachyon" field T with strength proportional to the height μ of the Fermi sea. The 0B string theory has two massless fields in its perturbative spectrum: the T field from the NSNS sector, and the RR pseudoscalar C. The observable of this theory is the string theory S-matrix S_{ST} describing scattering of T and C quanta off of the Liouville wall.

The purpose of this paper is to study this S-matrix of two-dimensional superstrings. S-matrices for two dimensional string theory in various backgrounds were also extensively studied in the previous incarnation of $c = 1$ matrix models. The work of Moore, Plesser and Ramgoolam (MPR) [21] determined the exact S-matrix for several variations on the

[1] Significant progress towards this discovery was made in [20], although that work fell short of identifying antisymmetric displacements of the Fermi sea as the fluctuating RR scalar of the 0B string theory.

[2] The linear dilaton solution given in [13] (2.3) is not a solution of the equations of motion coming from the effective action (2.4) for any functions $f_i(T)$. Neglected terms with higher derivatives in T will contribute at the same order due to the form of the tachyon profile.

inverted harmonic oscillator potential. One of these, which they called 'theory II', had a Fermi sea filled on both sides.[3] The S-matrix of MPR's theory II is essentially that of the 0B vacuum except for two differences, one minor and one major. The minor difference is that MPR interpreted fluctuations of the left and right Fermi seas as spacetime tachyons propagating in separate universes (speculatively related, for example, to the inside and outside of a black hole). In the 0B case we instead relate left and right fluctuations of the Fermi sea to the two separate spacetime fields [12, 13]

$$T_{\mathrm{L,R}} = \frac{1}{\sqrt{2}}(T \pm C)\,. \tag{1.1}$$

The major difference is that MPR found their S-matrix to be non-unitary in theory II, owing to the fact that their Hilbert space of asymptotic states did not include "soliton" sectors corresponding to single-fermion tunneling. We incorporate these explicitly, and find a unitary S-matrix.

There is also a type-0A theory in two dimensions. Unlike the 0B case, which has a unique vacuum, the 0A theory has a family of vacua labeled by an integer q which measures the D0-brane charge. There is only a single dynamical 'tachyon' field T in spacetime. The matrix model which has recently been proposed [13] as dual to the 0A theory was actually solved over a decade ago in a completely different guise [22]. Its S-matrices have fewer subtleties than that of 0B, so we will start by writing them down as a useful warm-up exercise.

The plan of this paper is as follows. In section 2 we review the central result of [21], that the spacetime S-matrix is essentially the matrix model S-matrix convolved with spacetime bosonization. This result applies readily to the 0A vacua. In section 3 we turn our attention to 0B and start by reviewing the apparent non-unitarity of the S-matrix of theory II, which arises from using an incomplete basis of in- and out- scattering states. Strict interpretation of the bosonization map dictates the addition of an infinite number of 'soliton' sectors labeled by an integer k which is dual to the zero mode of the RR field C. In section 4 we compute a number of simple amplitudes and determine the general structure of the S-matrix between arbitrary soliton sectors. Notably, we find that transition amplitudes between different soliton sectors are plagued by the usual infrared catastrophe of massless field theories. We propose a regularization in which the soliton sector vacua $|k\rangle$ are apparently not indistinguishable, rendering the diagonalization of the hamiltonian difficult. It would be interesting to see if there is an alternate regularization which treats the k-vacua on equal footing.

2. The S-matrix for two-dimensional string theory

2.1 The main formula

In string theory we are interested in the spacetime S-matrix for scattering T (or, in the 0B theory, T and C) off of the Liouville wall. In the matrix model we can easily calculate the

[3] The 'type-I' family of theories had an infinite potential wall added to prevent barrier penetration; from a modern perspective, this somewhat *ad hoc* prescription is less interesting.

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S-matrix for scattering free fermions off of the potential. In [21] it was shown that these two S-matrices are related to each other by the intuitive formula

$$S_{\mathrm{ST}} \sim S_{\mathrm{CF}} \equiv \imath_{f\to b} \circ S_{\mathrm{FF}} \circ \imath_{b\to f}\,, \tag{2.1}$$

where $\imath_{f\to b}$ is the bosonization map, $\imath_{b\to f}$ is its inverse and S_{CF} is known as the collective field S-matrix [23]. The symbol $\sim$ denotes an additional subtlety: the bosonized matrix model fermion is not identical to a spacetime field; rather the matrix model field and the spacetime field are related to each other by a particular background-dependent non-local field redefinition. Fortunately, the precise field redefinition appropriate for the 0A and 0B vacua is known, and will be incorporated below.

We stress that the bosonization in (2.1) is happening in spacetime, rather than on the worldsheet. Furthermore the formula (2.1) does not imply an exact relation between a non-relativistic fermion living in the matrix model potential and an object in the spacetime string field theory. Instead it only relates the asymptotics of correlation functions (i.e., S-matrix elements) of these two theories, and this only works because the spacetime theory is asymptotically a conformal field theory far from the Liouville wall [24]. In (2.1) it is understood that the fermionization map $\imath_{b\to f}$ acts only on modes coming in from $\mathcal{I}^-$, and the bosonization map $\imath_{f\to b}$ acts only on modes going out to $\mathcal{I}^+$.

2.2 The 0A S-matrix

We begin by applying the main formula (2.1) to the simpler case of the 0A theory. Essentially nothing in this section is actually new, since in fact the 0A theory was completely solved (unbeknownst to them, we presume) a decade ago by Demeterfi, Klebanov and Rodrigues [22]. The connection emerged only after the proposal of [13] that the 0A string theory in the background of q (stable) D0 branes is described by a $\mathrm{U}(N+q)\times\mathrm{U}(N)$ matrix model.[4] After taking the double scaling limit and integrating out the 'angular' variables of the matrix, the effective dynamics of the eigenvalues λ_i reduces to that of N free fermions on the half line $\lambda\in(0,\infty)$ with the hamiltonian

$$H = -\frac{1}{2}\frac{d^2}{d\lambda^2} - \frac{1}{2\alpha^2}\lambda^2 + \frac{q^2-\frac{1}{4}}{2\lambda^2}\,, \qquad \alpha\equiv\sqrt{2\alpha'}\,. \tag{2.2}$$

Such 'deformed' matrix models with a $1/\lambda^2$ potential in addition to the usual inverted harmonic oscillator term $-\lambda^2$ were extensively studied following [26].

Let us introduce canonically normalized fermionic mode operators $b(\omega)$. The empty Fermi sea $|0\rangle$ defined by $(H+\mu)|0\rangle$ satisfies

$$\begin{aligned} b(\omega)|0\rangle &= 0\,, \quad \omega > -\mu\,,\\ b^\dagger(\omega)|0\rangle &= 0\,, \quad \omega < -\mu\,. \end{aligned} \tag{2.3}$$

[4] The applicability of the DKR calculation to type-0A string theory was independently realized by Kapustin [18], who also pointed out a novel duality exchanging the string coupling and RR flux, previously noticed in the bosonic case by [25].

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Since the fermions are free, the S-matrix of the matrix model is completely determined in terms of the reflection coefficient for the hamiltonian (2.2), which was computed in [22]:

$$R(\omega) = \left|\frac{q^2 - \frac{1}{4} + \alpha^2\mu^2}{4}\right|^{-i\alpha\omega/2} \frac{\Gamma(\frac{1}{2} + \frac{|q|}{2} - \frac{i}{2}\alpha\mu + \frac{i}{2}\alpha\omega)}{\Gamma(\frac{1}{2} + \frac{|q|}{2} + \frac{i}{2}\alpha\mu - \frac{i}{2}\alpha\omega)} . \tag{2.4}$$

Overall phase factors independent of the frequency ω are irrelevant and have been dropped from (2.4). The physical content of the reflection coefficient (2.4) is that the unitary map between the in and out Hilbert spaces is simply

$$b_{\rm out}(\omega) = R(\omega) b_{\rm in}(\omega) = S b_{\rm in}(\omega) S^{-1} , \tag{2.5}$$

where the S-matrix is

$$S = \exp\left(- \int_{-\infty}^{\infty} d\omega \;\, \ln R(\omega) \;\, b_{\rm in}^\dagger(\omega) b_{\rm in}(\omega) \right) . \tag{2.6}$$

Next we wish to use (2.1) to express (2.6) in a Fock basis of massless in/out bosons rather than in/out fermions. To this end we introduce canonical bosonic modes $a(\omega)$ such that

$$[a(\omega), a(\omega')] = \omega \delta(\omega + \omega') \tag{2.7}$$

with $a^\dagger(\omega) = a(-\omega)$ and

$$a(\omega)|0\rangle = 0 , \quad \omega > 0 . \tag{2.8}$$

The fermionization map is given by

$$\imath_{b\to f} : \quad a(\omega) \to \int_{-\infty}^{\infty} d\xi \;\, b(\xi - \mu) b^\dagger(\xi - \omega - \mu) . \tag{2.9}$$

Note that μ enters in this formula in order to ensure that (2.8) is compatible with (2.3). Of course there are two separate copies of the fermionization map (2.9) which operate separately on the in/out Fock spaces.

The S-matrix for the scattering of $m \to n$ tachyons in the 0A theory is simply

$$S_{\rm CF}(\omega_1, \ldots, \omega_m | \omega'_1, \ldots, \omega'_n) \equiv \langle 0| \prod_{j=1}^{n} a_{\rm out}(\omega'_j) \prod_{k=1}^{m} a_{\rm in}(-\omega_k)|0\rangle , \tag{2.10}$$

where $a_{\rm out/in}$ are expressed in terms of $b_{\rm out/in}$ using (2.9). We use the convention that the arguments ω appearing in $S(\cdots|\cdots)$ are always positive. One can obtain explicit formulas for arbitrary amplitudes by substituting (2.9) and using (2.5). A set of graphical rules for immediately writing down any amplitude were presented in [21]. For example, the $1 \to n$ amplitude[5] is given by

$$\mathcal{A}_{\rm CF}(\omega|\omega_1, \ldots, \omega_n) = \sum_{k=0}^{n-1} (-1)^k \sum_{\{j_1,\ldots,j_k\}} \int_{\omega_{j_1}+\cdots+\omega_{j_k}}^{\omega} d\xi \;\, R(\omega - \xi) R^*(-\xi) \tag{2.11}$$

where $\omega = \omega_1 + \cdots + \omega_n$ and the final sum is over all possible length k subsets of $\{1, \ldots, n\}$.

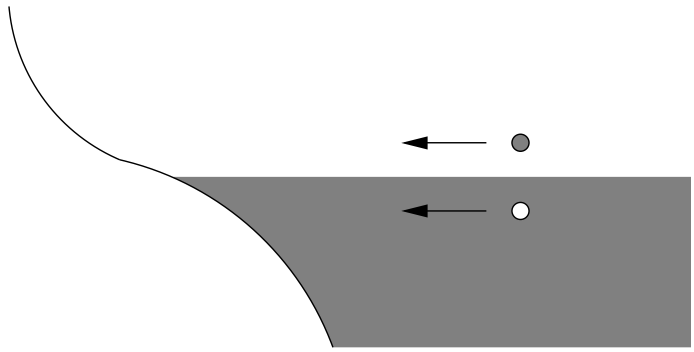

Figure 2: Scattering experiment in the 0A theory. Each incoming bosonic mode in the spacetime picture fermionizes into a particle/hole pair. These scatter individually off the potential with amplitudes (2.4) and return to infinity, where they are re-bosonized into some configuration of outgoing tachyons in spacetime.

Before proceeding, let us note that the picture shown in figure 2 depicts the case $q \neq 0$ so that the potential in (2.2) provides a repulsive barrier at small λ. However for $q = 0$ the potential is attractive near the origin. Nevertheless there is no bound state and the scattering matrix remains well-defined and unitary, at least for $\alpha\mu \neq \frac{1}{2}$. For $\alpha\mu = \frac{1}{2}$, problems appear to arise in the the formula (2.4) for the reflection coefficient. We have not investigated whether this is indicative of some interesting new strong coupling physics or whether it is just a coordinate singularity with no physical effects.

2.3 Three comments on the two-dimensional S-matrix

1. We have not yet taken into account the non-local transformation alluded to beneath (2.1). The spacetime tachyon field $T = T(t, \phi)$ is a function of time and the Liouville coordinate ϕ, whereas the field obtained by bosonizing the matrix model fermion is a function of time and the eigenvalue coordinate λ. The time coordinate t is identified between the matrix model and spacetime, but the coordinates ϕ and λ are related by a non-local transformation. This transformation is most easily written in momentum space, where it becomes a momentum-dependent phase factor (the so-called 'leg-pole factor')

$$e^{i\delta(\omega)} \equiv |q^2 - \tfrac{1}{4}|^{i\alpha\omega/4} \frac{\Gamma(-\frac{i}{2}\alpha\omega)}{\Gamma(\frac{i}{2}\alpha\omega)} \,. \tag{2.12}$$

   The string theory S-matrix is related to the $S_{\rm CF}$ in (2.10) by

$$S_{\rm ST}(\omega_1, \ldots, \omega_n | \omega_1', \ldots, \omega_m') = S_{\rm CF}(\omega_1, \ldots, \omega_n | \omega_1', \ldots, \omega_m') \prod_{i=1}^{n} e^{i\delta(\omega_i)} \prod_{j=1}^{m} e^{i\delta(\omega_j')} \,. \tag{2.13}$$

[5] We use the standard convention that $\mathcal{A}$ is equal to S with the overall energy conservation delta-function removed.

2. The result we wrote in (2.11) allows for particle production, yet when written in terms of fermions the theory is free and obviously cannot have any particle production! The resolution of this apparent puzzle lies in the fact that, as emphasized by MPR, a multi-boson state is not normal ordered with respect to the Fermi sea. Instead, we have (schematically)

$$\begin{aligned}\underbrace{a\cdots a}_{n}|0\rangle &= \underbrace{\int :b^\dagger b:\cdots\int :b^\dagger b:|0\rangle}_{n} \\ &=: \int \underbrace{bb^\dagger\cdots bb^\dagger}_{n} :|0\rangle + : \int \underbrace{bb^\dagger\cdots bb^\dagger}_{n-1} :|0\rangle + \cdots + : \int bb^\dagger :|0\rangle\,. \end{aligned} \tag{2.14}$$

Even though fermion number is strictly conserved, bosonic particle production can occur because the bosonization map relates a state with n bosons to a state of indefinite fermion number.

3. Finally, and most importantly, this S-matrix has no right to exist! We claim to have calculated an S-matrix for massless particles in two dimensions, but such an object suffers from two well-known pathologies. The first is that the S-matrix is generally ill-defined in the presence of massless particles due to infrared divergences associated with the proliferation of soft quanta. Consider the unitarity equation in the $1 \to 1$ channel:

$$1 = \frac{1}{\omega^2}|\mathcal{A}(\omega|\omega)|^2 + \frac{1}{\omega}\sum_{n=2}^{\infty}\prod_{i=1}^{n}\int_0^\infty \frac{d\omega_i}{\omega_i}\delta\left(\sum_{i=1}^{n}\omega_i - \omega\right)|\mathcal{A}(\omega|\omega_1,\ldots,\omega_n)|^2\,. \tag{2.15}$$

In a typical massless theory one needs to introduce an IR cutoff Λ in order to render S-matrix elements well-defined. Then the amplitude to emit any fixed number n of quanta with $\omega_i \approx 0$ vanishes linearly as $\Lambda \to 0$. However, these amplitudes cannot be dropped without violating unitarity because they contribute a finite amount to (2.15) after performing the ω integrals and summing over the number n of emitted soft photons. Essentially, an extra factor of Λ^{-2} comes from regulating the divergences in the density of states factor

$$\sum\prod\int_0^\infty \frac{d\omega_i}{\omega_i}\,. \tag{2.16}$$

In the present case, the amplitude to emit any fixed number of quanta is actually finite and non-zero without the need for an IR regulator. This is only consistent because all amplitudes vanish linearly as any one of the ω_i is taken to zero. For example,

$$\mathcal{A}_{\rm CF}(\omega|\omega) = \int_0^\omega d\xi\; R(\omega-\xi)R^*(-\xi) = \omega + \mathcal{O}(\omega^2)\,. \tag{2.17}$$

This fact renders all of the $\int \frac{d\omega}{\omega}$ integrals in (2.15) finite, so there is a finite contribution to the probability for any given number n of emitted quanta.

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This infrared catastrophe problem is a general feature of massless S-matrices in any dimension. In two dimensions there is yet another problem. Because of the special kinematics of massless particles in two dimensions, we cannot expect to tell the difference between, for example, a right-moving boson of energy ω and two right-moving bosons of energy $\omega/2$. This problem is solved in the present case by the fact that the S-matrix has a huge symmetry group related to W_∞ which is manifest in the fermionic formulation but obscure in the bosonic picture. One of the consequences of this large symmetry group is that it does allow us to distinguish these states [2].

3. The 0B S-matrix

Let us start by constructing the free fermion S-matrix $S_{\rm FF}$ for the double-filled Fermi sea. This essentially is the S-matrix for the so-called 'theory II' of Moore, Plesser and Ramgoolam [21]. The seeming non-unitarity of the resulting spacetime S-matrix $S_{\rm CF}$ will then be corrected by properly including the soliton sectors $|k\rangle$ implied by bosonization.

3.1 The free fermion S-matrix

Type-0B string theory in two dimensions has two physical fields in its perturbative spectrum: the spacetime 'tachyon' T (which is actually massless) and a massless RR scalar C. According to [12, 13] the dual matrix model involves free fermions with the familiar inverted harmonic oscillator hamiltonian

$$H = -\frac{1}{2}\frac{d^2}{d\lambda^2} - \frac{1}{2\alpha^2}\lambda^2\,, \tag{3.1}$$

and both sides of the Fermi sea filled. Fluctuations of $T_{\rm L,R} \equiv \frac{1}{\sqrt{2}}(T \pm C)$ are identified with fluctuations of the left and right Fermi seas. Clearly $T_{\rm L}$ and $T_{\rm R}$ are perturbatively decoupled and only interact with each other by non-perturbative tunneling of fermions through the barrier.

Let us start by writing down the free fermion S-matrix. At this stage the only new ingredient as compared to the previous 0A discussion is that we now have twice as many fermionic modes, which can be incident from either the left or the right. The second quantized fermion field $\hat\psi(t,\lambda)$ is now expanded as

$$\hat\psi(t,\lambda) = \int_{-\infty}^{\infty} d\omega\; e^{i\omega t}\left[b_{\rm L}(\omega)\chi_{\rm L}(\omega,\lambda) + b_{\rm R}(\omega)\chi_{\rm R}(\omega,\lambda)\right], \tag{3.2}$$

where $\chi_{\rm L/R}(\omega,\lambda)$ are a complete set of normalized wavefunctions of the hamiltonian (3.1) describing fluctuations incident from the left or right, and the mode operators satisfy

$$\{b_\epsilon(\omega), b^\dagger_\eta(\omega')\} = \delta_{\epsilon\eta}\delta(\omega-\omega')\,, \quad \epsilon,\eta \in \{L,R\}\,. \tag{3.3}$$

The empty Fermi sea is defined by (2.3) separately for both sets of oscillators.

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A single excitation of the Fermi sea sent in from the left or right can either scatter off the barrier or tunnel through it. The transmission and reflection coefficients were calculated in [21]:

$$R(\omega) = \frac{1}{\sqrt{2\pi}}(i\alpha\mu)^{i\alpha\omega}\Gamma\left(\frac{1}{2} + i\alpha\mu - i\alpha\omega\right), \qquad T(\omega) = -ie^{\pi\alpha(\omega-\mu)}R(\omega)\,. \tag{3.4}$$

Note that

$$|R(\omega)|^2 = \frac{1}{1+e^{-2\pi\alpha(\mu-\omega)}}\,, \qquad |T(\omega)|^2 = \frac{1}{1+e^{+2\pi\alpha(\omega-\mu)}} \tag{3.5}$$

and

$$|R|^2 + |T|^2 = 1\,, \qquad RT^* + TR^* = 0\,. \tag{3.6}$$

The in and out Hilbert spaces are related by the unitary transformation

$$\begin{pmatrix} b_{\rm L}^{\rm out} \\ b_{\rm R}^{\rm out} \end{pmatrix} = \begin{pmatrix} R & T \\ T & R \end{pmatrix} \begin{pmatrix} b_{\rm L}^{\rm in} \\ b_{\rm R}^{\rm in} \end{pmatrix}. \tag{3.7}$$

Since L and R are exchanged under $\lambda \to -\lambda$, we may define a basis

$$b_\pm \equiv \frac{1}{\sqrt{2}}(b_{\rm L} \pm b_{\rm R})\,, \tag{3.8}$$

associated to wavefunctions that are manifestly symmetric or antisymmetric.[6] The unitary transformation (3.7) becomes diagonal in the $\pm$ basis,

$$b_\pm^{\rm out}(\omega) = R^\pm(\omega) b_\pm^{\rm in}(\omega)\,, \tag{3.9}$$

where

$$R^\pm(\omega) = R(\omega) \pm T(\omega)\,. \tag{3.10}$$

We conclude that the free-fermion S matrix exactly factorizes into two commuting pieces:

$$S = S_+S_-\,, \qquad S_\pm = \exp\left(-\int_{-\infty}^{\infty} d\omega\ \ln R^\pm(\omega)\ b_{\pm\rm in}^\dagger(\omega) b_{\pm\rm in}(\omega)\right). \tag{3.11}$$

In the 0B theory we combine the two spacetime fields T, C into $T_{\rm L,R}$, which are separately fermionized into $b_{\rm L,R}$. We caution the reader against concluding from (1.1) and (3.8) that the factorization of the fermionic S-matrix into $S = S_+S_-$ implies that the bosonic S-matrix factorizes into $S = S_T S_C$. This factorization would imply that the T and C fields do not talk to each other! This does not happen because the modes $b_\pm$ are not simply related to those of T and C because bosonization is non-linear:

$$\begin{aligned} \frac{1}{\sqrt{2}}(a_{\rm L}(\omega) \pm a_{\rm R}(\omega)) &= \frac{1}{\sqrt{2}}\int_{-\infty}^{\infty} d\xi \left[b_{\rm L}(\xi-\mu) b_{\rm L}^\dagger(\xi-\omega-\mu) \pm ({\rm L} \leftrightarrow {\rm R})\right] \\ &\neq \int_{-\infty}^{\infty} d\xi\ b_\pm(\xi-\mu) b_\pm^\dagger(\xi-\omega-\mu)\,. \end{aligned} \tag{3.12}$$

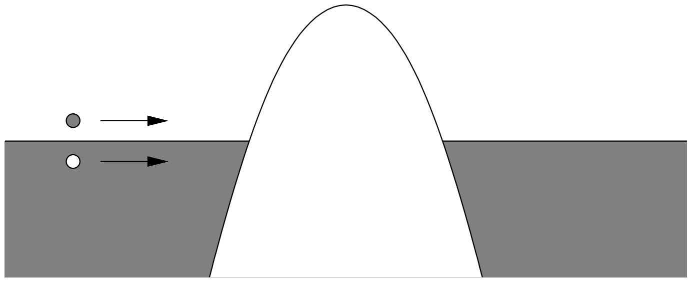

Figure 3: This figure shows the matrix model description of the process where we send in a single quantum of T_L from infinity towards the Liouville wall.

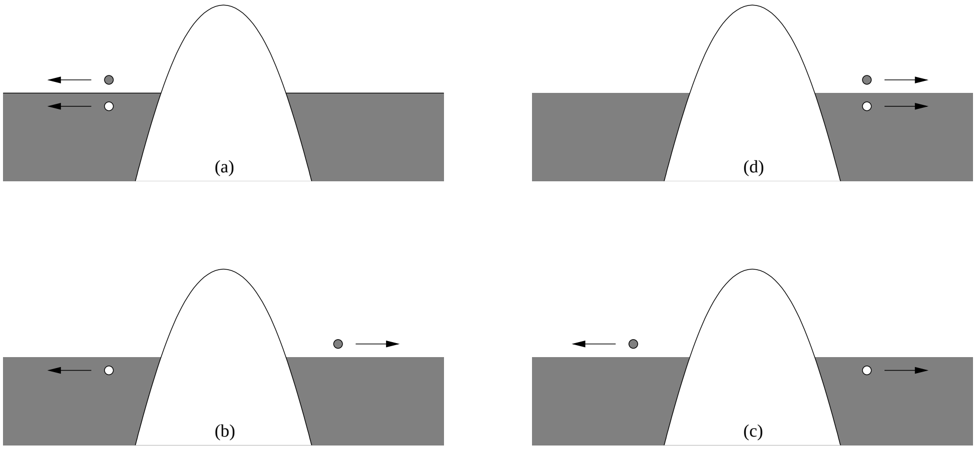

Figure 4: The four possible final states of the experiment depicted in figure 3, labeled (a), (b), (c), and (d) in order of decreasing probability.

3.2 The problem with 'theory-II'

Let us consider a scattering experiment where we prepare a single quantum of the T_L field and send it in towards the Liouville wall. This bosonizes into the particle/hole pair shown in figure 3.

Since the fermions are free and the total fermion number is conserved, there are obviously exactly four possible final states for this process, shown in figure 4. We have labeled the final states in order of decreasing probability. Note that it is always easier for a particle to tunnel through the barrier than it is for a hole to tunnel, since the barrier is always narrower for the particle. When μ is large, (b) and (c) are suppressed relative to (a) by a factor of $e^{-\pi\alpha\mu}$ and (d) is suppressed by a factor of $e^{-2\pi\alpha\mu}$.

The spacetime interpretation of (a) and (d) is clear: they correspond respectively to a single quantum of T_L or T_R emerging from the Liouville wall and going out to infinity. The spacetime interpretation of (b) and (c) is more subtle. In fact these processes were not given any spacetime interpretation in [21], and as a result their S-matrix was considered not unitary.

[6] Note that while the $\chi_{\mathrm{L/R}}$ in (3.2) coincide with wavefunctions in (A.3) of [21], the basis $\chi_\pm$ defined by (3.8) differs from the $\psi^\pm$ in (A.1) and (A.2) of [21]. Furthermore, note that the $\pm$ superscript on $\chi^\pm_{\mathrm{L/R}}$ of that paper distinguishes 'in' and 'out', while the $\pm$ subscript on our $\chi_\pm$ denotes even or odd parity.

The philosophy of this paper is that the fundamental formula (2.1) should be taken as literally true. Since the free fermion S-matrix $S_{\rm FF}$ is manifestly unitary, the failure of $S_{\rm ST}$ to be unitarity must necessarily be due to a failure in the bosonization map used in the previous subsection. This failure is nothing more than the familiar fact that finite energy excitations of T and C on the vacuum $|0\rangle$ do not give a complete basis for the image of the bosonization map. Of course the S-matrix cannot be unitary if an incomplete basis of in- and out- scattering states is used [27]. In the next subsection we review the construction of the soliton sectors necessary to provide a complete basis.

3.3 Bosonization

We follow the conventions of [28]. Consider the holomorphic part of a compact boson X given by the mode expansion

$$\partial X(z) = -i \sum_m \frac{a_m}{z^{m+1}}\,, \quad a_m = \frac{1}{2\pi} \oint dz\; z^m \partial X(z)\,. \tag{3.13}$$

The modes satisfy

$$[a_m, a_n] = m\delta_{m,-n}\,. \tag{3.14}$$

The Hilbert space splits into disconnected sectors $\mathcal{H}_k$, where k labels the zero mode momentum and each $\mathcal{H}_k$ is a Fock space built on the state $|k\rangle$ satisfying

$$a_m|n\rangle = 0 \quad \forall m > 0\,, \qquad a_0|k\rangle = k|k\rangle\,. \tag{3.15}$$

Bosonization relates ∂X to a complex fermion ψ according to

$$i\partial X =: \psi\psi^\dagger :\,. \tag{3.16}$$

If we write ψ as

$$\psi(z) = \sum_{r\in\mathbb{Z}+\frac{1}{2}} z^{-r-1/2} b_r\,, \tag{3.17}$$

where

$$\{b_r, b_s^\dagger\} = \delta_{rs} \tag{3.18}$$

and the vacuum $|0\rangle$ satisfies

$$b_r|0\rangle = 0 \quad \forall r > 0\,, \qquad b_s^\dagger|0\rangle = 0 \quad \forall s < 0\,, \tag{3.19}$$

then (3.15) implies that

$$a_n = \sum_{r\in\mathbb{Z}+\frac{1}{2}} : b_r b_{r-n}^\dagger :\,. \tag{3.20}$$

In particular, for $n = 0$ we find

$$a_0 = \sum_{r\in\mathbb{Z}+\frac{1}{2}} : b_r b_r^\dagger := \sum_{r<0} b_r b_r^\dagger - \sum_{r>0} b_r^\dagger b_r\,, \tag{3.21}$$

which is just the fermion number operator (#particles − #holes).

The hamiltonian is

$$L_0 = \frac{1}{2}a_0^2 + \sum_{n>0} a_{-n}a_n = \sum_{r>0} r(b_{-r}b^\dagger_{-r} - b^\dagger_r b_r)\,. \tag{3.22}$$

The state $|k\rangle$ may be identified in the fermionic picture as the unique state with fermion number $a_0|k\rangle = k|k\rangle$ and energy $L_0|k\rangle = \frac{k^2}{2}|k\rangle$:

$$\begin{aligned} |k\rangle &= b_{-k+\frac{1}{2}} \cdots b_{-\frac{1}{2}}|0\rangle\,, \quad k>0\,, \\ |k\rangle &= b^\dagger_{-k-\frac{1}{2}} \cdots b^\dagger_{\frac{1}{2}}|0\rangle\,, \quad k<0\,. \end{aligned} \tag{3.23}$$

When the holomorphic and antiholomorphic pieces of $X(z)$ are combined, $|k\rangle$ can be written in the familiar way in the bosonized theory as a soliton,

$$|k\rangle = e^{ikX}(0)|0\rangle\,. \tag{3.24}$$

For this reason we will sometimes refer to $|k\rangle$ as the charge-k soliton.

It is useful to consider an operator $\mathcal{O}$ which satisfies

$$[a_m, \mathcal{O}] = \delta_{m,0}\mathcal{O}\,. \tag{3.25}$$

It follows that $\mathcal{O}$ is the raising operator conjugate to a_0, in that

$$\mathcal{O}|k\rangle = |k{+}1\rangle\,. \tag{3.26}$$

It is easy to work out how the action $\mathcal{O}$ is represented on the fermionic modes,

$$\mathcal{O}b_r = b_{r-1}\mathcal{O}\,. \tag{3.27}$$

This operator is of course nothing other than the spectral flow operator [28] (not from the NS to R sector, but from NS past R back to NS again). We remind the reader again that this spectral flow operator is acting in spacetime, not on the worldsheet.

The crucial lesson is that the full fermionic Hilbert space built by acting with the creation operators b_r and $b^\dagger_s$, for $r<0$ and $s>0$, is equivalent not to a single bosonic Fock space, but to an infinite family of Fock spaces. Each bosonic Fock space has a ground state $|k\rangle$ where k is dual to the net fermion number. Bosonization is a unitary map

$$\imath_{f\to b} : \mathcal{H} \to \bigoplus_{k=-\infty}^{\infty} \mathcal{H}_k\,. \tag{3.28}$$

When we wrote down the 0A S-matrices in the previous section, however, we did not worry about the different $\mathcal{H}_k$ sectors. The reason is that fermion number is strictly conserved in the matrix model, so it is perfectly consistent to work in a superselection sector with fixed fermion number (equal to zero, if we normal order with respect to the Fermi sea).

In the 0B theory, on the other hand, we have two fermions $\psi_{\rm L,R}$ which bosonize separately into $T_{\rm L,R}$. Therefore the vacuum state should be labeled by two zero-mode indices $|k_{\rm L}, k_{\rm R}\rangle$. The total fermion number $k_{\rm L}+k_{\rm R}$ is still exactly conserved, so just as in the 0A

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case we can consistently work in a superselection sector with fixed $k_{\rm L}+k_{\rm R}=0$. However, the relative fermion number $k\equiv\frac{1}{2}(k_{\rm L}-k_{\rm R})$ is not conserved and is in general changed when fermions tunnel through the potential. Note that k is related to the zero mode of the C collective field:[7]

$$C_0|k\rangle=\frac{1}{\sqrt{2}}(T_{\rm L,0}-T_{\rm R,0})|k\rangle=\frac{1}{\sqrt{2}}(k_{\rm L}-k_{\rm R})|k\rangle=\sqrt{2}k|k\rangle. \tag{3.29}$$

It follows from (3.23) that the ground state in the charge k sector is

$$|k\rangle=b_{{\rm L},-k+\frac{1}{2}}\cdots b_{{\rm L},-\frac{1}{2}}b^\dagger_{{\rm R},k-\frac{1}{2}}\cdots b^\dagger_{{\rm R},\frac{1}{2}}|0\rangle=e^{i\sqrt{2}kC}(0)|0\rangle. \tag{3.30}$$

As written this formula is for $k>0$; for $k<0$ one interchanges L and R.

In summary, the S-matrix of theory II failed to be unitary because the basis of asymptotic scattering states used in the calculation was not complete. The complete Hilbert space is $\bigoplus\mathcal{H}_k$, and unitarity is manifest if we allow states built on arbitrary $|k\rangle$ to appear in the S-matrix.

3.4 Discretization and the definition of the k-vacua

In the previous subsection we used fields with discrete mode expansions, rather than the continuous mode expansions we used in section 2 for the 0A theory. We will see below that the 0B theory, unlike 0A, unfortunately does suffer from the infrared problems one usually encounters in massless theories. The essence of the problem is that the solitons $|k\rangle$ are coherent states built out of quanta with zero energy. In order to tame the infrared catastrophe, we will follow [27] and introduce an infrared cutoff by making time periodic with period $2\pi/\beta$. This cutoff restricts all energies to be multiples of β. States then carry discrete indices as in the previous subsection, according to the dictionary

$$b_r\leftrightarrow b(r\beta)\,,\qquad a_n\leftrightarrow a(n\beta)\,. \tag{3.31}$$

At the end of any calculation involving some modes of the form (3.31), we take $\beta\to 0, r, n\to\infty$ keeping βr, βn fixed. Note that we put antiperiodic boundary conditions on the Fermi fields in compactified time.

The exception to the latter scaling is that the index k of the ground states $|k\rangle$ is held fixed in the continuum limit, so

$$|k\rangle=\lim_{\beta\to 0}b_{\rm L}((-k+\frac{1}{2})\beta)\cdots b_{\rm L}(-\frac{1}{2}\beta)b^\dagger_{\rm R}((k-\frac{1}{2})\beta)\cdots b^\dagger_{\rm R}(\frac{1}{2}\beta)|0\rangle\,. \tag{3.32}$$

Therefore the states $|k\rangle$ become degenerate (at zero energy) in the continuum limit, so we will refer to them as the k-vacua. Intuitively, $|k\rangle$ may be thought of as the state where k particles are skimmed from the surface of the right Fermi sea and gently lowered onto the surface of the left Fermi sea.

[7]The notation C_0 means 'the zero mode of ∂C', in the same sense that a_0 is the zero mode of ∂X in (3.13). We point this out only to highlight that it is *not* the zero mode of C itself.

This intuitive picture is made concrete by generalizing the operator $\mathcal{O}$ defined in (3.25), (3.26) and (3.27) to the present case of the double-filled Fermi sea. Essentially we now want $\mathcal{O} = \mathcal{O}_{\rm L}\mathcal{O}_{\rm R}^\dagger$, which satisfies

$$\begin{aligned} \mathcal{O}|k\rangle &= |k{+}1\rangle\,, \\ [a_{{\rm L},m},\mathcal{O}] &= \delta_{m,0}\mathcal{O}\,, \qquad [a_{{\rm R},m},\mathcal{O}] = -\delta_{m,0}\mathcal{O}\,, \\ \mathcal{O}b_{{\rm L},r} &= b_{{\rm L},r-\beta}\mathcal{O}\,, \qquad \mathcal{O}b_{{\rm R},r} = b_{{\rm R},r+\beta}\mathcal{O}\,. \end{aligned} \tag{3.33}$$

The last line makes manifest the fact that the raising operator $\mathcal{O}$ amongst the k-vacua works in the continuum limit $\beta \to 0$, βr fixed by raising the left Fermi sea and lowering the right Fermi sea by infinitesimal amounts (see figure 5).

These states may seem indistinguishable, in the continuum limit, from the 'bare' Fermi sea $|0\rangle$, but it is essential to keep track of them in order to get a unitary S-matrix. In particular, the configurations in figure 4b and c are orthogonal to all states in the Fock space built on $|0\rangle$!

3.5 The k-vacua in spacetime: D-instanton charge

We have claimed that the spectrum of scattering states in 0B string theory is incomplete without including the "solitonic" $\mathcal{H}_k$ sectors. Here we discuss further the spacetime interpretation of these sectors. Given that the k-changing transitions in the matrix model correspond to eigenvalue tunneling, which as reviewed in the introduction is a process of magnitude $e^{-\mathcal{O}(1/g)}$ in spacetime, it should come as no surprise that the $\mathcal{H}_k$ sectors are related to the physics of D-instantons.

The effective action for the 0B fields possesses, to all orders in string perturbation theory, a shift symmetry in the RR scalar C [13],

$$\mathcal{T}_a: \quad C \to C + a\,. \tag{3.34}$$

Such a symmetry, or its higher-rank p-form generalization, is generic for Ramond-Ramond fields. It implies the existence of a Noether current J_μ, and as is the case in ten-dimensional IIB string theory, the objects possessing the RR scalar charge are the D-instantons. The D-instanton number n in a spacetime process can be evaluated by integrating over a circle at infinity,

$$n = \int_{S^1} dx^\mu J_\mu\,. \tag{3.35}$$

For example, in the expression given for the leading-order 0B effective action in [13], the RR scalar appears as

$$S_{\rm 0B} \supset -\frac{1}{8\pi}\int d^2x\sqrt{-g}f_3(T)(\nabla C)^2\,, \tag{3.36}$$

where $f_3(T)$ is a function of the tachyon field identified there as $f_3(T) = e^{-2T}$. The Noether current is then

$$J_\mu = f_3(T)\partial_\mu C\,, \tag{3.37}$$

where we assumed a flat background metric, and (3.35) by Stokes theorem is equivalently expressed as an integral over all spacetime of a density,

$$n = \int dJ = \int f_3'(T)\, dT \wedge dC\,. \tag{3.38}$$

This expression is the 2D analog of the Pontryagin density $\int F \wedge F$ for 4D Yang-Mills instantons.

Consider now the symmetry $\mathcal{T}_a$ acting on the k-vacua. Using (3.30) , we find[8]

$$\mathcal{T}_a : \quad |k\rangle \to e^{i\sqrt{2}ka}|k\rangle\,. \tag{3.39}$$

States in the matrix model with nonvanishing difference between left and right fermion numbers (relative to the undisturbed Fermi seas) are therefore charged under $\mathcal{T}_a$. The existence of the k-vacua breaks the shift symmetry of C non-perturbatively, though a residual identification renders the field compact:

$$C \sim C + \sqrt{2}\pi\,. \tag{3.40}$$

The transformation (3.39) indicates that the k-vacua have nonzero RR charge. Hence for any state in the sector $\mathcal{H}_k$, we must have

$$\int d\phi\, J_0 = k \tag{3.41}$$

at some moment in time, where we remind the reader that ϕ is the spacelike Liouville coordinate. A D-instanton with charge n (3.35) can then mediate classically forbidden transitions between a state with charge k and one of charge $n+k$, in accordance with our intuition based on the $e^{-\mathcal{O}(1/g)}$ strength of single-eigenvalue tunneling. The k-vacua, then, have a relationship to D-instantons precisely analogous to that in Yang-Mills theory of the "winding vacua" to Yang-Mills instantons.

In the matrix model description, we found that the k-vacua are degenerate in energy. This is again closely analogous to the Yang-Mills case. In order to dominate the path integral, instantons must have finite action. A consequence of this is that the action must vanish at infinity where the charge (3.35) is measured; in Yang-Mills theory, this requires $F_{\mu\nu} = 0$ asymptotically. But the integral defining the charge k of the state (3.41) is identical to that of the fields at infinity around a charge-k instanton.[9] In Yang-Mills theory, this implies the existence of a state of each charge k with vanishing field strength and hence zero energy. Although we lack an adequate effective action for the spacetime fields, it seems probable that the finiteness of the action for configurations of k D-instantons also implies the existence of zero energy states of charge k, namely the k-vacua.

[8]We have not forgotten the nonlocal transformation advertised beneath (2.1) between the collective field used to define (3.30) and the spacetime field of 0B string theory. However, it can be ignored for the zero mode of the field, since as discussed in the next section, the appropriate leg pole factor is unity at zero energy.

[9]This is true when the fields in (3.41) approach the same value at infinity, effectively compactifying the spacelike slice.

It would be highly interesting to find the correct effective action for 0B string theory and construct a D-instanton solution. If the correct action is some iteration of (2.4) in [13], it is clear that more than simply the C-field will be involved, as any nonzero value for the RR current (3.37) will backreact on the other fields.

Because of the D-instanton processes, the k-vacua will mix, and will not be eigenvectors of the full hamiltonian. It is then natural to inquire what the energy eigenstates are. In the Yang-Mills case, the winding vacua are related by a large gauge transformation and hence possess precisely equivalent physics. The diagonalization of the hamiltonian then proceeds analogously to the study of Bloch waves in quantum mechanics: the eigenvectors of the hamiltonian are a band of θ-vacua, and the energy density of the states goes like $\cos\theta$. Similarly, one may define for 0B string theory a set of c-vacua:

$$|c\rangle \equiv \sum_k e^{-i\sqrt{2}kc}|k\rangle\ . \tag{3.42}$$

These states are conjugate to the k-vacua, and are eigenvectors of the zero mode of C. θ-vacua in Yang-Mills theory are superselection sectors; we expect however that in our two-dimensional case, transitions between the c-vacua can be effected by sending in energy in the form of C or T quanta.

The analogy to Yang-Mills physics persists only if the k-vacua prove to be indistinguishable. We have already argued in the matrix model that the k-vacua are energetically degenerate; we have not, however, shown they have equivalent physics. In the next section, we calculate a few basic amplitudes and provide evidence that the sectors of different k are, in fact, different with our choice of infrared regularization. If this turns out to be the case, the diagonalization of the hamiltonian will be more involved.

4. Some 0B amplitudes

In this section we use the prescription of the previous section to calculate several interesting amplitudes in the 0B theory. These examples will highlight how easy it is to calculate arbitrary non-perturbative S-matrix elements as exact functions of μ. Furthermore we will verify our claim that the usual infrared catastrophe of massless theories is present. In general amplitudes must be calculated in the discretized theory at $\beta > 0$, where the theory is manifestly unitary. In the limit $\beta \to 0$ only probabilities are in general well-defined.

4.1 Leg-pole factors

Before proceeding we should note that the leg-pole factors which relate the collective field S-matrix $S_{\rm CF}$ to the string theory S-matrix $S_{\rm ST}$ were obtained in [13] and are given by

$$e^{i\delta_T(\omega)} = \frac{\Gamma(\frac{i}{2}\alpha\omega)}{\Gamma(-\frac{i}{2}\alpha\omega)}\,, \qquad e^{i\delta_C(\omega)} = \frac{\Gamma(\frac{1}{2}+\frac{i}{2}\alpha\omega)}{\Gamma(\frac{1}{2}-\frac{i}{2}\alpha\omega)} \tag{4.1}$$

for the NSNS and RR fields respectively. It has been emphasized [29, 30, 3] that these factors are essential for obtaining a local theory of gravity in two spacetime dimensions. However, since most of our calculations below involve simple transition probabilities, we will consider the collective field description to be adequate and neglect the leg pole factors. Note that the leg-pole factors go to 1 at $\omega = 0$, so these factors do not affect the zero energy particles/holes constituting the k-vacua.

4.2 Transitions between k-vacua

We start with very explicit calculations of the simplest amplitudes. Using the definition (3.30) we consider for $k_1, k_2 \geq 0$ the amplitude

$$\langle k_1|S|k_2\rangle = \langle 0|b^{\rm out}_{{\rm R},\frac{1}{2}} \cdots b^{\rm out}_{{\rm R},k_1-\frac{1}{2}} b^{\rm out\dagger}_{{\rm L},-\frac{1}{2}} \cdots b^{\rm out\dagger}_{{\rm L},-k_1+\frac{1}{2}} b^{\rm in}_{{\rm L},-k_2+\frac{1}{2}} \cdots b^{\rm in}_{{\rm L},-\frac{1}{2}} b^{\rm in\dagger}_{{\rm R},k_2-\frac{1}{2}} \cdots b^{\rm in\dagger}_{{\rm R},\frac{1}{2}}|0\rangle\,. \tag{4.2}$$

Next we use (3.7) to relate $b^{\rm out}$ to $b^{\rm in}$. Consider the left-most oscillator $b^{\rm out}_{{\rm R},\frac{1}{2}}$. The only oscillator it can possibly contract with is $b^{\rm in\dagger}_{{\rm R},\frac{1}{2}}$ all the way on the right. This gives a factor of $R(\frac{1}{2}\beta)$. Following this logic gives a unique set of non-zero contractions which leads to

$$\langle k_1|S|k_2\rangle = \delta_{k_1,k_2} \prod_{j=1}^{k_1} R\left(\left(j-\frac{1}{2}\right)\beta\right) R^*\left(\left(-j+\frac{1}{2}\right)\beta\right). \tag{4.3}$$

Next consider $k_1, -k_2 \geq 0$. In this case we simply exchange the L and R subscripts on all of the $b^{\rm in}$'s in (4.2). The contractions then give the transmission coefficient T instead of R. Therefore (4.3) is extended to all integers k_1, k_2 by writing

$$\begin{aligned}\langle k_1|S|k_2\rangle = {} & \delta_{k_1,k_2} \prod_{j=1}^{|k|} R\left(\left(j-\frac{1}{2}\right)\beta\right) R^*\left(\left(-j+\frac{1}{2}\right)\beta\right) + \\ & +\delta_{k_1,-k_2} \prod_{j=1}^{|k|} T\left(\left(j-\frac{1}{2}\right)\beta\right) T^*\left(\left(-j+\frac{1}{2}\right)\beta\right).\end{aligned} \tag{4.4}$$

This is the complete answer in the discretized theory.

In the continuum limit all of the arguments of R and T go to zero, and we are left with

$$\langle k_1|S|k_2\rangle = \frac{\delta_{k_1,k_2}}{(1+e^{-2\pi\alpha\mu})^{|k|}} + \frac{\delta_{k_1,-k_2}}{(1+e^{+2\pi\alpha\mu})^{|k|}}\,, \tag{4.5}$$

where we used (3.5).

We went through this calculation in exquisite detail simply to show how the procedure works, but of course the answer (4.5) could have been written down immediately after looking at figure 5. First, it is clear that $|k\rangle$ can only mix with $|\pm k\rangle$ (in the discretized theory) by energy conservation. The first term in (4.5) corresponds to $|2k|$ particles and holes reflecting off of the barrier, while in the second term all of the particles and holes tunnel through the barrier, causing $|k\rangle$ to evolve into $|-k\rangle$.

Our first indication that the 0B theory suffers classic infrared problems is that the S-matrix (4.5) is not unitary when restricted to the k-vacua. This is because there are infinitely many additional states whose energies vanish in the continuum limit and which

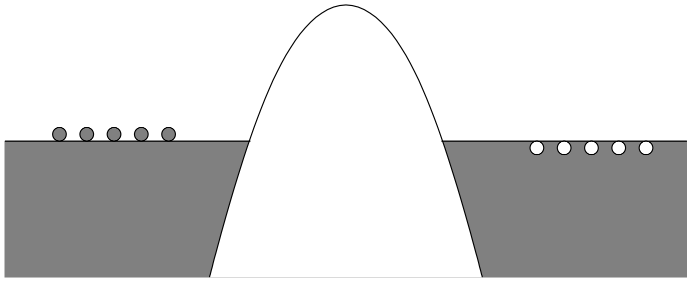

Figure 5: To construct the $|k\rangle$ vacuum one skims k particles from top of the right Fermi sea and deposits them on top of the left Fermi sea (or vice versa, for $k < 0$).

mix with the k-vacua. In particular, the initial state $|1\rangle$ can evolve to the following four final states:

$$\begin{aligned}
\langle 1|S|1\rangle &= R\left(\frac{1}{2}\beta\right) R^*\left(-\frac{1}{2}\beta\right) = \frac{1}{1+e^{-2\pi\alpha\mu}}, \\
\langle -1|S|1\rangle &= T\left(\frac{1}{2}\beta\right) T^*\left(-\frac{1}{2}\beta\right) = \frac{1}{1+e^{+2\pi\alpha\mu}}, \\
\langle 0|a_{\mathrm{L},1}S|1\rangle &= R\left(\frac{1}{2}\beta\right) T^*\left(-\frac{1}{2}\beta\right) = i\frac{e^{-\pi\alpha\mu}}{1+e^{-2\pi\alpha\mu}}, \\
\langle 0|a_{\mathrm{R},1}S|1\rangle &= T\left(\frac{1}{2}\beta\right) R^*\left(-\frac{1}{2}\beta\right) = -i\frac{e^{-\pi\alpha\mu}}{1+e^{-2\pi\alpha\mu}},
\end{aligned} \tag{4.6}$$

where in the final expressions we have taken $\beta \to 0$ inside the arguments. These are of course nothing more than our old friends from figure 4! Of course if we square the amplitudes in (4.6) and sum them up we find that the unitarity condition for $|1\rangle$ is satisfied:

$$1 = \sum_{|\alpha\rangle} |\langle\alpha|S|1\rangle|^2 . \tag{4.7}$$

In the continuum limit, we leave the value $k = 1$ for the initial state fixed, and as a consequence the bosons $a_{\mathrm{L},1}$, $a_{\mathrm{R},1}$ in the last two processes become soft quanta with vanishing energy. Just like in QED, a finite fraction of the probability is lost to states with additional excitations of arbitrarily low energy. The S-matrix is therefore not well-defined in the continuum limit and we must content ourselves with transition probabilities.

Let us therefore calculate the total probability for the initial state $|k\rangle$ to evolve into any state of charge l. This kind of calculation would be very difficult in the bosonic formulation because we would have to sum over final states containing arbitrarily large numbers of T_{L} and T_{R} quanta. Fortunately the calculation is absolutely trivial in the free fermion representation. First note that since the energy of $|k\rangle$ is $\frac{1}{2}\beta k^2$, energy conservation requires $|l| \leq |k|$. Next note that the fermionic S-matrix (3.11) never mixes b with $b^\dagger$ and never touches the mode index—all it can do is switch L and R with probability $\approx |T(0)|^2$ (again setting $\beta = 0$ inside the argument). Therefore, if we start with

$$|\mathrm{in}\rangle = |k\rangle = b_{\mathrm{L},-k+\frac{1}{2}} \cdots b_{\mathrm{L},-\frac{1}{2}} b^\dagger_{\mathrm{R},k-\frac{1}{2}} \cdots b^\dagger_{\mathrm{R},\frac{1}{2}} |0\rangle , \tag{4.8}$$

then there are only $4^{|k|}$ possible out states given by

$$|\mathrm{out}\rangle = |k\rangle = b_{\mathrm{F}_1,-k+\frac{1}{2}} \cdots b_{\mathrm{F}_k,-\frac{1}{2}} b^\dagger_{\mathrm{G}_1,k-\frac{1}{2}} \cdots b^\dagger_{\mathrm{G}_k,\frac{1}{2}} |0\rangle \,, \tag{4.9}$$

where $\{F_i, G_i\} \in \{\mathrm{L}, \mathrm{R}\}$. Furthermore, if we recall the definition of the charge

$$C_0|\mathrm{out}\rangle = \sqrt{2}m = \frac{1}{\sqrt{2}}(\#b_{\mathrm{L}} - \#b^\dagger_{\mathrm{L}} - \#b_{\mathrm{R}} + \#b^\dagger_{\mathrm{R}}) \,, \tag{4.10}$$

then there are precisely $\binom{2|k|}{|k+m|}$ different out states for any fixed value of $m \in [-k, k]$. Finally note that the probability $|\langle \mathrm{out}|\mathrm{in}\rangle|^2$ is actually independent of the particular values of F_i, G_i in the continuum limit (because all energies ω are taken to the Fermi sea). Therefore, we conclude that the transition probability from the $|k\rangle$-vacuum to the charge m sector is

$$P(|k\rangle \to \mathcal{H}_m) = \sum_{|\alpha\rangle \in \mathcal{H}_m} |\langle \alpha|S|k\rangle|^2 = \begin{pmatrix} 2|k| \\ |k+m| \end{pmatrix} \frac{e^{-2\pi\alpha\mu|k-m|}}{(1+e^{-2\pi\alpha\mu})^{2|k|}} \,. \tag{4.11}$$

Again, this result immediately follows from figure 5. The combinatorial factor $\binom{2|k|}{|k+m|}$ counts how many distinct ways there are for $|k-m|$ particles and holes to tunnel through the barrier. This factor multiplies the transition probability $|R(0)|^{2|k+m|}|T(0)|^{2|k-m|}$. We conclude this analysis by noting that the unitarity condition

$$1 = \sum_{m=-\infty}^{\infty} P(|k\rangle \to \mathcal{H}_m) \qquad \forall k \in \mathbb{Z} \tag{4.12}$$

is of course satisfied by (4.11).

The transition probabilities we calculated in (4.11) are notably undemocratic in the continuum limit. In particular, $|k\rangle$ can only decay to those $|m\rangle$ with $|m| \leq |k|$. Of course the origin of this fact is that in the discretized theory, the energy of $|k\rangle$ is equal to βk^2 so no process can increase $|k|$. At first though one might have expected the probabilities (4.11) to depend on β in such a way that in the $\beta \to 0$ limit, $P(|k\rangle \to \mathcal{H}_m)$ would depend only on $|k-m|$ and democracy between the k-vacua is restored. However, the order of limits that we take prevents this from happening. It would be interesting to determine if there is an alternate regularization procedure which does give democratic amplitudes in the continuum limit.

4.3 $\mathcal{H}_0 \to \mathcal{H}_0$ amplitudes

Next we consider amplitudes for processes with no incoming or outgoing solitons. This section is brief because in this subspace, the S-matrix is well defined, even in the continuum theory. For example, the simplest $1 \to 1$ S-matrix elements are

$$\begin{aligned} \langle 0|a_{\mathrm{L}}(\omega')Sa_{\mathrm{L}}(-\omega)|0\rangle &= \langle 0|a_{\mathrm{R}}(\omega')Sa_{\mathrm{R}}(-\omega)|0\rangle = \delta(\omega-\omega')\int_0^\omega d\xi\; R(\omega-\xi)R^*(-\xi)\,, \\ \langle 0|a_{\mathrm{L}}(\omega')Sa_{\mathrm{R}}(-\omega)|0\rangle &= \langle 0|a_{\mathrm{R}}(\omega')Sa_{\mathrm{L}}(-\omega)|0\rangle = \delta(\omega-\omega')\int_0^\omega d\xi\; T(\omega-\xi)T^*(-\xi)\,. \end{aligned} \tag{4.13}$$

Discretization is necessary for the $|k \neq 0\rangle$ vacua because finite probability is carried away by states with zero energy, but this does not happen for $\mathcal{H}_0 \to \mathcal{H}_0$ amplitudes for the same reason it did not happen in the 0A theory.

In [31], where the Liouville theory on the string worldsheet was studied, it was shown that correlation functions of the vertex operators $\mathcal{V}_{\mathrm{L,R}}$ (corresponding respectively to the spacetime fields $T_{\mathrm{L,R}}$) factorize at tree level:

$$\begin{aligned} \left\langle \prod_{i=1}^{n} \mathcal{V}_{\mathrm{L}}(k_i) \prod_{j=1}^{m} \mathcal{V}_{\mathrm{R}}(k_j) \right\rangle &= 0 \,, \quad (n, m \geq 1) \,, \\ \left\langle \prod_{i=1}^{n} \mathcal{V}_{\mathrm{L}}(k_i) \right\rangle = \left\langle \prod_{i=1}^{n} \mathcal{V}_{\mathrm{R}}(k_i) \right\rangle &= \frac{1}{4} \left\langle \prod_{i=1}^{n} \mathcal{V}(\sqrt{2} k_i) \right\rangle_{\text{bosonic}} \,, \end{aligned} \tag{4.14}$$

where $\mathcal{V}$ is the tachyon vertex operator in the bosonic Liouville theory. The structure of the exact S-matrix in the zero soliton sector implies that a weaker form of (4.14) continues to hold at arbitrary order in string perturbation theory. Namely, the Liouville theory vertex operator correlation functions continue to satisfy (4.14) at arbitrary genus after one integrates over the moduli space.

4.4 Some $\mathcal{H}_0 \to \mathcal{H}_1$ amplitudes

Let us consider the amplitude for a simple one-particle state $|\mathrm{in}\rangle \equiv a_{\mathrm{L}}(\omega)|0\rangle$ to scatter into the 1 soliton sector. In the discretized theory, we write $a_{\mathrm{L}}(\omega)|0\rangle = a_{\mathrm{L},n}|0\rangle$, with the understanding that we will take $n \to \infty$ as $\beta \to 0$ with $\omega = n\beta$ fixed.

The subtleties inherent to massless fields are immediately apparent when we try to identify possible final states for this process. The most obvious candidate is $|\mathrm{out}\rangle = a_{\mathrm{L}}(\omega)|1\rangle = a_{\mathrm{L},n}|1\rangle$, but this cannot mix with $|\mathrm{in}\rangle$ in the discretized theory because of energy conservation! Instead we are forced to consider $a_{\mathrm{L},n-1}|1\rangle$, which is essentially indistinguishable from $a_{\mathrm{L}}(\omega)|1\rangle$ in the continuum limit.

The S-matrix element is

$$\langle 1| a_{\mathrm{L},n'-1} S a_{\mathrm{L},-n} |0\rangle = T\left(\frac{1}{2}\beta\right) R^*\left(\left(\frac{1}{2} - n\right)\beta\right) \delta_{n,n'} \,. \tag{4.15}$$

In the continuum limit this gives

$$\langle 1| a_{\mathrm{L}}(\omega') S a_{\mathrm{L}}(-\omega) |0\rangle = \beta \delta(\omega - \omega') T(0) R^*(-\omega) \,, \tag{4.16}$$

where the factor of β comes from writing $\delta_{n,n'} = \beta\delta(\omega - \omega')$. The overall factor of β naively implies that this amplitude can be dropped relative to $\mathcal{H}_0 \to \mathcal{H}_0$ S-matrix

$$\langle 0| a_{\mathrm{L}}(\omega') S a_{\mathrm{L}}(-\omega) |0\rangle = \delta(\omega - \omega') \int_0^{\omega} d\xi \; R(\omega - \xi) R^*(-\xi) \tag{4.17}$$

in the continuum limit. This conclusion is wrong, however, for reasons that are familiar by now: there are infinitely many final states in the charge 1 sector which give amplitudes that seemingly vanish in the $\beta \to 0$ limit. Examples include

$$a_{\mathrm{L},-n+1} a_{\mathrm{L},-1}|1\rangle \,, \quad a_{\mathrm{L},-n+7} a_{\mathrm{R},-3} a_{\mathrm{L},-4}|1\rangle \,, \quad \text{etc.}, \tag{4.18}$$

where, in the continuum limit, $n \to \infty$ but the indices on the additional quanta are held fixed, rendering them soft. If we were powerful enough to calculate all of these amplitudes and sum of their squares, we would find a finite probability for $a_{\rm L}(\omega)$ to decay into the charge 1 sector. Fortunately, the power of the free fermion representation actually makes this calculation trivial! The result is

$$\begin{aligned}
P(a_{\rm L}(-\omega)|0\rangle \to \mathcal{H}_0) &= \frac{1}{\omega}\int_0^\omega d\xi\ |R(\omega-\xi)R^*(-\xi)|^2 + |T(\omega-\xi)T^*(-\xi)|^2\,, \\
P(a_{\rm L}(-\omega)|0\rangle \to \mathcal{H}_1) &= \frac{1}{\omega}\int_0^\omega d\xi\ |T(\omega-\xi)R^*(-\xi)|^2\,, \\
P(a_{\rm L}(-\omega)|0\rangle \to \mathcal{H}_{-1}) &= \frac{1}{\omega}\int_0^\infty d\xi\ |R(\omega-\xi)T^*(-\xi)|^2\,.
\end{aligned} \tag{4.19}$$

Naturally these probabilities add up to one. The first line is a sum of finite contributions, one of which comes from the square of (4.17). However the second and third lines come from summing over an infinite number of amplitudes such as (4.16) which individually are all zero but in total contribute a finite probability. We note as a curiosity that it is actually possible to perform the integrals (4.19) analytically.

4.5 Some $\mathcal{H}_1 \to \mathcal{H}_1$ amplitudes

Next we consider

$$\langle 1|a_{\rm L}(\omega')Sa_{\rm L}(-\omega)|1\rangle = \delta(\omega-\omega')|R(0)|^2\int_0^\omega d\xi\ R(\omega-\xi)R^*(-\xi) + \mathcal{O}(\beta)\,. \tag{4.20}$$

The factor of $|R(0)|^2$ comes from contracting the incoming and outgoing $b^\dagger_{{\rm R},\frac{1}{2}}$ which appears in the definition of $|1\rangle$. Terms in which the $b_{{\rm L},-\frac{1}{2}}$ in $|1\rangle$ contract with one of the fermions in $a_{\rm L}(\omega)$ are subleading in the continuum limit.

In this case it is consistent to drop the additional $\mathcal{O}(\beta)$ piece. It cannot possibly contribute a finite amount to transition probabilities, since if it did, then the leading term in (4.20) would contribute an infinite amount of probability, thereby violating unitarity!

4.6 General structure of the S-matrix

The examples we investigated in the previous subsections illustrate the general structure of the S-matrix. Decomposing S into the various soliton sectors, we find

$$S \sim \begin{pmatrix} & & \vdots & & \\ & |R(0)|^2S_0 & \mathcal{O}(\beta) & |T(0)|^2S_0 & \\ \cdots & \mathcal{O}(\beta) & S_0 & \mathcal{O}(\beta) & \cdots \\ & |T(0)|^2S_0 & \mathcal{O}(\beta) & |R(0)|^2S_0 & \\ & & \vdots & & \end{pmatrix}. \tag{4.21}$$

Here S_0 is the perfectly well-defined (although non-unitary), β-independent S-matrix for scattering in the charge 0 sector. In other words, S_0 is just the 'theory II' S-matrix of [21]. The formula (4.21) demonstrates schematically that (1) the S-matrix is rendered unitary by including solitonic sectors labeled by integers k dual to the zero mode of the RR scalar C, and that (2) as is familiar in massless theories, unitarity forbids us from dropping amplitudes which naively vanish as the IR cutoff β is taken to zero, since these amplitudes can make finite contributions to probabilities.

4.7 Decaying D-branes

The 0B string theory has unstable D0-branes which have been interpreted [12, 13] in the matrix model as a single fermion perched at the top of the potential (see figure 6).

Of course these D-branes don't appear as external states in our S-matrix since they are not stable, but we can consider a process where we send in a very finely tuned initial state specially designed to kick a single fermion up to the top of the potential. A state which does the trick is

$$|D\rangle = b_{\mathrm{L},-\hat{\mu}} b^{\dagger}_{\mathrm{L},-\frac{1}{2}} |0\rangle \,. \tag{4.22}$$

Following our standard conventions, we will keep $\mu = \hat{\mu}/\beta$ fixed in the continuum limit. This state takes a very complicated form when expressed in terms of bosonic quanta, indicating the large extent to which it is fine-tuned.

Clearly the possible decay products have charge ± 1 or 0. A calculation similar to several we have already done yields the probabilities

$$\begin{aligned} P(|D\rangle \to \mathcal{H}_0) &= \frac{1}{2} \,, \\ P(|D\rangle \to \mathcal{H}_{-1}) &= \frac{1}{2} \frac{1}{1 + e^{-2\pi\alpha\mu}} \,, \\ P(|D\rangle \to \mathcal{H}_1) &= \frac{1}{2} \frac{1}{1 + e^{+2\pi\alpha\mu}} \,. \end{aligned} \tag{4.23}$$

Of course these results are completely in line with the physical picture of figure 6.

It is straightforward to calculate S-matrix elements for $|D\rangle$ decay into any particular state in the charge 0 sector. However S-matrix elements into the charge ± 1 sectors will suffer from the usual IR problems.

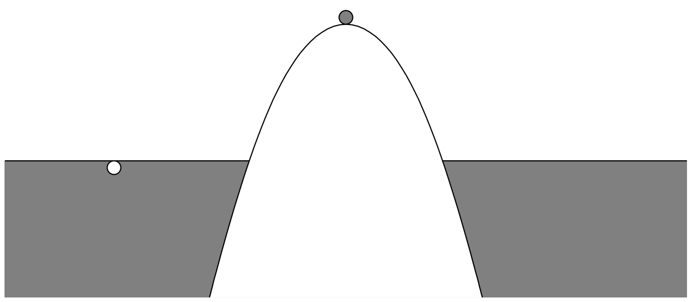

Figure 6: Matrix model description of a single unstable 0B D-brane. The extra hole is present because we have decreed that we are working in a superselection sector where the total fermion number is fixed to zero (relative to the total Fermi sea).

5. Conclusion and discussion

We have obtained prescriptions for calculating the exact, non-perturbative S-matrices for the "linear dilaton" vacua of two-dimensional 0A and 0B string theory. The underlying assumptions are that the proposed string/matrix model dualities [12, 13] are correct even non-perturbatively, and that the string theory S-matrix factorizes into the leg-pole factors (4.1) times the free-fermion S-matrix convolved with bosonization, as in (2.1). We have shown that in the 0B theory a crucial role is played by the RR-charged "soliton" sectors $|k\rangle$, and that infrared problems require special attention and prevent in some cases the definition of amplitudes, forcing one instead to work with probabilities.

There are a number of interesting problems to which this technology can be applied. One application is the diagonalization of the hamiltonian, as was alluded to in the text. The straightforward expectation is that the eigenstates are the "theta-vacua" for the states $|k\rangle$, namely the eigenvectors $|c\rangle$ of the RR scalar zero mode, with a potential $\cos c$. However this is only obtained straightforwardly if the k-vacua are indistinguishable, something the calculations we have presented argue against. One may ask whether an alternate regularization procedure than the one presented here might produce democratic amplitudes. Further investigation to obtain the correct eigenvectors would naturally be very interesting.

Generalizing these methods to other string theory backgrounds is of course also of interest. In particular, the two-dimensional type-II superstring has been studied in [17]; it is claimed to preserve a cubic potential, rather than a quadratic one, in the double scaling limit. One may consider scattering in this background using our methods; the Fermi theory will no longer be trivial due to the more complicated potential. The well around the minimum may have an interpretation as a black hole [32].

Another application of S-matrix calculations is to determine a low-energy effective action. One certainly may find solutions of zero-dimensional string theory in the free fermion picture by just making 'blobs' of Fermi sea and letting them evolve through phase space in time, but an effective action provides a bridge to a familiar spacetime interpretation. However, as discussed in the introduction, at present we do not possess even a lowest-order solution to a lowest-order effective action around which to start expanding. The S-matrices are background-dependent and are only useful once the background around which one is scattering is established. It would be very useful to determine this lowest-order, linear-dilaton-type solution.

A non-perturbative description of a string theory also allows one to address questions about gravitational physics, such as black holes. Unstable phenomena such as unstable branes or black holes that form and Hawking decay should appear as resonances in the scattering matrix. We have discussed briefly how to engineer a state that appears as an unstable D-brane at intermediate times; it would be fascinating to improve our knowledge of brane and gravitational physics by studying further the possible resonances.

Finally, with the exact S-matrix in hand it will be possible to explore quantitatively the strong coupling behavior of string theory in these two dimensional type 0 toy models (see for instance [18] for interesting work in this direction). A tantalizing speculation concerns

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the existence of a three-dimensional '0M' theory as strong coupling dual of the 0A theory, whose field content can be matched to that of 3d gravity coupled to a scalar field.

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